PERGAMON Computers and Mathematics with Applications 38 (1999) 37–43

An International Journal
computers & mathematics
with applications

Stability Criteria for a Class of Linear Delay Partial Difference Equations

B. G. ZHANG
Department of Applied Mathematics
Ocean University of Qingdao
Qingdao 266003, P.R. China

CHUAN JUN TIAN
Department of Electronic and Communication Engineering
South China University of Technology
Guangzhou, 510641, P.R. China

(Received June 1998; revised and accepted October 1998)

Abstract—This paper is concerned with the linear delay partial difference equation

$$a(i,j)u(i+1,j)+u(i,j+1)+b(i,j)u(i,j)+p(i,j)u(i-\sigma,j-\tau)=0,$$

where $\{a(i,j)\},\{b(i,j)\},\{p(i,j)\},\ i,j\in N_0$, are real sequences. Sufficient conditions for this equation to be stable are derived. Some conditions for this equation to be unstable are obtained also.

Keywords—Partial difference equations, Stable, Unstable.

1. INTRODUCTION

Stability of ordinary difference equations has been investigated extensively, see [1, Chapter 5; 2–5]. In particular, the global attractivity of solutions of the difference equation

$$x_{n+1}-x_n+p_nx_{n-k}=0$$

has been studied in [3].

We consider the partial difference equation

$$a(i,j)u(i+1,j)+u(i,j+1)+b(i,j)u(i,j)+p(i,j)u(i-\sigma,j-\tau)=0, \tag{1}$$

where $\{a(i,j)\},\{b(i,j)\},\{p(i,j)\},\ i,j\in N_0$, are real sequences, $\sigma,\tau\in N_0,\ N_i=\{i,i+1,\cdots\}$.

This work was supported by NNSF of China.
We thank the referee for some good suggestions.

PII: S0898-1221(99)00180-7

Such equation arises as difference schemes for calculating approximate solutions of certain partial differential equations [6,7]. Also, this equation arises from consideration of random walk problems [8, p. 28], mathematical physics problem [9], and the study of molecular orbits [10].

A solution of (1) is a real double sequence $u = \{u(i,j)\}$, which satisfies (1).

Let $\Omega = N_{-\sigma} \times N_{-\tau} \setminus N_0 \times N_1$. Given a function $\phi(i,j)$ defined on Ω, it is easy to construct by induction a double sequence $\{u(i,j)\}$, which equals $\phi(i,j)$ on Ω and satisfies (1) on $N_0 \times N_1$. In fact, we rewrite (1) in the form

$$u(i,j+1) = -a(i,j)u(i+1,j) - b(i,j)u(i,j) - p(i,j)u(i-\sigma, j-\tau).$$

Then we can calculate successively $u(0,1); u(1,1), u(0,2); u(2,1), u(1,2), u(0,3); \dots$. Such a double sequence is unique and is said to be a solution of (1) with the initial condition

$$u(i,j) = \phi(i,j), \qquad (i,j) \in \Omega.$$

Oscillation of (1) has been investigated in [11–14]. For the special case that $a(i,j) \equiv 1$, $b(i,j) \equiv -1$, we have proved that (1) has no the global attractivity of solutions [15], i.e., we always can find a solution $\{u(i,j)\}$ of (1), which does not tend to zero as $i \to \infty$, $j \to \infty$. This property of (1) is different from the mentioned result [3] for ordinary difference equations. Recently, stability results of some special cases of (1) have been obtained in [8,16].

In this paper, we will study the local stability of (1). For any $H > 0$,

$$S_H = \left\{ \phi \mid \sup_{(i,j)\in\Omega} |\phi(i,j)| < H \right\}.$$

The following definitions are similar to those for ordinary difference equations [2].

DEFINITION 1. *Equation (1) is said to be stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $\phi \in S_\delta$, the corresponding solution $u = \{u(i,j)\}$ of (1) satisfies*

$$|u(i,j)| < \epsilon, \qquad i,j \in N_0. \tag{2}$$

DEFINITION 2. *Equation (1) is said to be exponentially asymptotically stable if for any $\delta > 0$, there exists a real number $\xi \in (0,1)$ such that $\phi \in S_\delta$ implies that*

$$|u(i,j)| \le \delta \xi^j, \qquad i,j \in N_0. \tag{3}$$

2. MAIN RESULTS

THEOREM 1. *Assume that*

$$|a(i,j)| + |b(i,j)| + |p(i,j)| \le 1, \qquad i,j \in N_0. \tag{4}$$

Then equation (1) is stable.

PROOF. For any $\epsilon > 0$, let $\delta = \epsilon$ and $\phi \in S_\delta$. We claim that the solution satisfies (2). In fact, from (1) we have

$$\begin{aligned} u(i,1) &= -a(i,0)u(i+1,0) - b(i,0)u(i,0) - p(i,0)u(i-\sigma,-\tau) \\ &= -a(i,0)\phi(i+1,0) - b(i,0)\phi(i,0) - p(i,0)\phi(i-\sigma,\tau), \qquad i \ge 0. \end{aligned}$$

Hence,

$$|u(i,1)| < \epsilon[|a(i,0)| + |b(i,0)| + |p(i,0)|] \le \epsilon, \qquad i \ge 0.$$

Assume that for some fixed integer $n \geq 0$,

$$|u(i,j)| < \epsilon, \qquad \text{for } i \geq -\sigma, \quad -\tau \leq j \leq n.$$

Then from (1), we have

$$u(i,n+1) = -a(i,n)u(i+1,n) - b(i,n)u(i,n) - p(i,n)u(i-\sigma, n-\tau).$$

Hence,

$$|u(i,n+1)| < \epsilon\left(|a(i,n)| + |b(i,n)| + |p(i,n)|\right) \leq \epsilon, \qquad i \geq 0.$$

By induction, we obtain

$$|u(i,j)| < \epsilon, \qquad i,j \geq 0.$$

The proof is complete.

THEOREM 2. *Assume that $\sigma = 0$ and there exists a real number $\xi \in (0,1)$ such that*

$$|a(i,j)| + |b(i,j)| + \xi^{-\tau}|p(i,j)| \leq \xi, \qquad i,j \in N_0. \tag{5}$$

Then (1) is exponentially asymptotically stable.

PROOF. Let $\delta > 0$ and $\phi \in S_\delta$. From (1), we have

$$\begin{aligned} u(i,1) &= -a(i,0)u(i+1,0) - b(i,0)u(i,0) - p(i,0)u(i,-\tau) \\ &= -a(i,0)\phi(i+1,0) - b(i,0)\phi(i,0) - p(i,0)\phi(i,-\tau), \qquad i \geq 0, \end{aligned}$$

which implies

$$|u(i,1)| \leq \delta\left(|a(i,0)| + |b(i,0)| + |p(i,0)|\right) \leq \delta\xi, \qquad i \in N_0.$$

In general, for $0 \leq j \leq \tau$, we have

$$|u(i,j)| \leq \delta\xi^j, \qquad i \in N_0, \quad 0 \leq j \leq \tau.$$

Assume that for some fixed integer $n \geq \tau$,

$$|u(i,j)| \leq \delta\xi^j, \qquad \text{for } i \geq 0, \quad 0 \leq j \leq n.$$

Then from (1), we have

$$\begin{aligned} |u(i,n+1)| &= |-a(i,n)u(i+1,n) - b(i,n)u(i,n) - p(i,n)u(i,n-\tau)| \\ &\leq \delta\xi^n|a(i,n)| + \delta\xi^n|b(i,n)| + \delta\xi^{n-\tau}|p(i,n)| \\ &= \delta\xi^n(|a(i,n)| + |b(i,n)| + \xi^{-\tau}|p(i,n)|) \leq \delta\xi^{n+1}, \qquad i \geq 0. \end{aligned}$$

By induction, we have

$$|u(i,j)| \leq \delta\xi^j, \qquad \text{for } i,j \in N_0.$$

Hence, (1) is exponentially asymptotically stable. The proof is complete.

COROLLARY 1. *If $\sigma = 0$, $p(i,j) \equiv p$ for all $i,j \in N_0$ and there exists a real number $\xi \in (0,1)$ such that*

$$|a(i,j)| + |b(i,j)| + |p| \leq \xi, \qquad i,j \in N_0, \tag{6}$$

then (1) is exponentially asymptotically stable.

Now we will show sufficient conditions for (1) to be unstable.

THEOREM 3. *Assume that one of the following conditions holds.*

(i) $a(i,j) \geq 0$, $b(i,j) \geq 0$, $p(i,j) \geq 0$ *for* $i,j \in N_0$, τ *is even and there exists a real number* $r > 1$ *such that*

$$a(i,j) + b(i,j) \geq r, \qquad i,j \in N_0. \tag{7}$$

(ii) $a(i,j) \geq 0$, $b(i,j) \geq 0$, *and* $p(i,j) \leq 0$, τ *is odd and (7) holds.*

(iii) $a(i,j) \leq 0$, $b(i,j) \geq 0$, *and* $p(i,j) \geq 0$, $\sigma + \tau$ *is even, and*

$$-a(i,j) + b(i,j) \geq r > 1, \qquad i,j \in N_0. \tag{8}$$

(iv) $a(i,j) \leq 0$, $b(i,j) \geq 0$, *and* $p(i,j) \leq 0$, $\sigma + \tau$ *is odd and (8) holds.*

(v) $a(i,j) \geq 0$, $b(i,j) \leq 0$, *and* $p(i,j) \geq 0$, σ *is odd, and*

$$a(i,j) - b(i,j) \geq r > 1, \qquad i,j \in N_0. \tag{9}$$

(vi) $a(i,j) \geq 0$, $b(i,j) \leq 0$, *and* $p(i,j) \leq 0$, σ *is even and (9) holds.*

Then (1) is unstable.

PROOF. We only give the proof for cases (1), (iii), and (v). The other cases can be proved by the similar method.

If (i) holds, we take $\phi(i,j) = (-1)^j \delta$, $(i,j) \in \Omega$. From (1), we have

$$\begin{aligned} u(i,1) &= -a(i,0)u(i+1,0) - b(i,0)u(i,0) - p(i,0)u(i-\sigma,-\tau) \\ &= -\delta(a(i,0) + b(i,0) + p(i,0)) < 0, \qquad i \in N_0. \end{aligned}$$

Hence,

$$|u(i,1)| = \delta(a(i,0) + b(i,0) + p(i,0)) \geq \delta(a(i,0) + b(i,0)) \geq \delta r, \qquad i \in N_0.$$

It is easy to see that $u(i,2) > 0$ and $|u(i,2)| \geq \delta r^2$, $i \in N_0$. Assume that for some fixed integer $n \geq 0$,

$$(-1)^j u(i,j) > 0, \qquad \text{for } i \geq -\sigma \text{ and } -\tau \leq j \leq n,$$

and

$$|u(i,j)| \geq \delta r^j \qquad \text{for } i \geq 0, \quad 0 \leq j \leq n.$$

Then from (1), for all $i \geq 0$, we have

$$\begin{aligned} (-1)^{n+1} u(i,n+1) &= -(-1)^{n+1} \left(a(i,n)u(i+1,n) + b(i,n)u(i,n) + p(i,n)u(i-\sigma,n-\tau) \right) \\ &= a(i,n) \left((-1)^n u(i+1,n) \right) + b(i,n) \left((-1)^n u(i,n) \right) \\ &\quad + p(i,n) \left((-1)^{n-\tau} u(i-\sigma,n-\tau) \right) > 0. \end{aligned}$$

Hence,

$$\begin{aligned} |u(i,n+1)| &= a(i,n)|u(i+1,n)| + b(i,n)|u(i,n)| \\ &\quad + p(i,n)|u(i-\sigma,n-\tau)| \geq \delta r^n (a(i,n) + b(i,n)) \geq \delta r^{n+1}, \qquad i \in N_0. \end{aligned}$$

By induction, we have

$$|u(i,j)| \geq \delta r^j, \qquad i,j \in N_0.$$

Clearly, $|u(i,j)| \to \infty$ as $j \to \infty$, then (1) is unstable.

If (iii) holds, we take $\phi(i,j) = (-1)^{i+j}\delta$, $(i,j) \in \Omega$. From (1), we have

$$\begin{aligned} u(i,1) &= -a(i,0)u(i+1,0) - b(i,0)u(i,0) - p(i,0)u(i-\sigma,-\tau) \\ &= (-1)^{i+1}\delta(-a(i,0) + b(i,0) + p(i,0)), \qquad i \in N_0. \end{aligned}$$

Hence, $(-1)^{i+1}u(i,1) > 0$ for $i \geq 0$, and

$$|u(i,1)| = \delta(-a(i,0) + b(i,0) + p(i,0)) \geq \delta r, \qquad i \geq 0.$$

Assume that for some fixed integer $n \geq 0$,

$$\begin{aligned} (-1)^{i+j}u(i,j) &> 0, \qquad \text{for } i \geq -\sigma, \quad -\tau \leq j \leq n, \\ |u(i,j)| &\geq \delta r^j, \qquad \text{for } i \geq 0, \qquad 0 \leq j \leq n. \end{aligned}$$

Then from (1), we obtain

$$\begin{aligned} (-1)^{i+n+1}u(i,n+1) = {} & -a(i,n)\left[(-1)^{i+n+1}u(i+1,n)\right] + b(i,n)\left[(1)^{i+n}u(i,n)\right] \\ & + p(i,n)\left[(-1)^{i-\sigma+n-\tau}u(i-\sigma,n-\tau)\right] > 0, \qquad i \geq 0, \end{aligned}$$

which implies

$$\begin{aligned} |u(i,n+1)| &= -a(i,n)|u(i+1,n)| + b(i,n)|u(i,n)| + p(i,n)|u(i-\sigma,n-\tau)| \\ &\geq \delta r^n(-a(i,n) + b(i,n)) \geq \delta r^{n+1}, \qquad i \geq 0. \end{aligned}$$

By induction, we obtain

$$|u(i,j)| \geq \delta r^j, \qquad i,j \in N_0.$$

Then (1) is unstable.

If (v) holds, we take $\phi(i,j) = (-1)^i\delta$, $(i,j) \in \Omega$. From (1), we have

$$\begin{aligned} u(i,1) &= -a(i,0)u(i+1,0) - b(i,0)u(i,0) - p(i,0)u(i-\sigma,-\tau) \\ &= (-1)^i\delta(a(i,0) - b(i,0) + p(i,0)), \qquad i \geq 0. \end{aligned}$$

Hence, $(-1)^iu(i,1) > 0$ for $i \geq 0$, and

$$|u(i,1)| = \delta(a(i,0) - b(i,0) + p(i,0)) \geq \delta r, \qquad i \geq 0.$$

Assume that for some fixed integer $n \geq 1$,

$$\begin{aligned} (-1)^iu(i,j) &> 0, \qquad \text{for } i \geq -\sigma, \quad -\tau \leq j \leq n, \\ |u(i,j)| &\geq \delta r^j, \qquad \text{for } i \geq 0, \qquad 0 \leq j \leq n. \end{aligned}$$

Then from (1), we obtain

$$\begin{aligned} (-1)^iu(i,n+1) = {} & a(i,n)\left[(-1)^{i+1}u(i+1,n)\right] - b(i,n)\left[(-1)^iu(i,n)\right] \\ & + p(i,n)\left[(-1)^{i-\sigma}u(i-\sigma,n-\tau)\right] > 0, \qquad i \in N_0, \end{aligned}$$

and

$$\begin{aligned} |u(i,n+1)| &= a(i,n)|u(i+1,n)| - b(i,n)|u(i,n)| + p(i,n)|u(i-\sigma,n-\tau)| \\ &\geq \delta r^n(a(i,n) - b(i,n)) \geq \delta r^{n+1}, \qquad i \in N_0. \end{aligned}$$

By induction, we have

$$|u(i,j)| \geq \delta r^j, \qquad i,j \in N_0.$$

Hence (1) is unstable. The proof is complete.

REMARK 1. Under conditions of Theorem 3, we have proved that there exists at least an unbounded solution of (1).

REMARK 2. In Theorem 3 (i), if $p(i,j) \geq p > 0$, then the conclusion of Theorem 3 is valid for $r \geq 1$. In fact, in the above proof we can obtain

$$|u(i,j)| \geq \delta r^j + p\delta, \qquad i,j \in N_0.$$

Therefore (1) is unstable even $r = 1$.

REMARK 3. We compare condition (4) (for stability) and condition (7) (for unstability) to find that there is a gap between (4) and (7). How do we fill this gap? That is an open problem.

Similarly, we can prove the following result.

THEOREM 4. *Assume that $\sigma = 0$. If one of the following conditions holds.*

(i) $a(i,j), b(i,j), p(i,j) \geq 0$ *for* $i,j \geq 0, \tau$ *is even and there exists a real number* $r > 1$ *such that*

$$a(i,j) + b(i,j) + r^{-\tau} p(i,j) \geq r, \qquad i,j \in N_0. \tag{10}$$

(ii) $a(i,j) \geq 0,\ b(i,j) \geq 0,\ p(i,j) \leq 0,\ \tau$ *is odd, and*

$$a(i,j) + b(i,j) - r^{-\tau} p(i,j) \geq r > 1, \qquad \text{for } i,j \in N_0. \tag{11}$$

(iii) $a(i,j) \leq 0,\ b(i,j) \geq 0,\ p(i,j) \geq 0,\ \tau$ *is even, and*

$$-a(i,j) + b(i,j) + r^{-\tau} p(i,j) \geq r > 1, \qquad i,j \in N_0. \tag{12}$$

(iv) $a(i,j) \leq 0,\ b(i,j) \geq 0,\ p(i,j) \leq 0,\ \tau$ *is odd, and*

$$-a(i,j) + b(i,j) - r^{-\tau} p(i,j) \geq r > 1, \qquad i,j \in N_0. \tag{13}$$

(v) $a(i,j) \geq 0,\ b(i,j) \leq 0,\ p(i,j) \leq 0,$ *and*

$$a(i,j) - b(i,j) - r^{-\tau} p(i,j) \geq r > 1, \qquad i,j \in N_0. \tag{14}$$

Then (1) is unstable.

EXAMPLE 1. Consider the partial difference equation

$$\frac{1}{2}u(i+1,j) + u(i,j+1) - \frac{1}{2}u(i,j) = 0, \qquad i,j \in N_0. \tag{15}$$

By Theorem 1, (15) is stable. In fact, $u(i,j) = 2^{-(i+2j)},\ i,j \in N_0$ is a solution of (15).

EXAMPLE 2. Consider the equation

$$u(i+1,j) + u(i,j+1) + bu(i,j) = 0, \qquad i,j \in N_0. \tag{16}$$

From Theorem 3, if $b > 0$, then (16) is unstable. In fact, if $b > 0$, (16) has an unbounded solution $u(i,j) = (-1-b)^j,\ i,j \in N_0$.

EXAMPLE 3. Consider the difference equation with variable coefficients

$$\frac{1}{16}u(i+1,j) + u(i,j+1) + \frac{1}{4}\left(\frac{17}{8} - (i+j+1)^{-1}\right)u(i,j) + [16(i+j+1)]^{-1}u(i,j-2) = 0. \tag{17}$$

We take $\xi = 31/32$, then condition (5) holds. By Theorem 2, (17) is exponentially asymptotically stable. In fact,

$$u(i,j) = \delta 2^{-(i+j)}, \qquad i,j \in N_0$$

is such a solution of (17), which satisfies (3).

REFERENCES

1. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
2. S. Elaydi and S. Zhang, Stability and periodicity of difference equations with finite delay, *Funkcial Ekvac* **37**, 401–413, (1994).
3. L.H. Erbe, H. Xia and J.S. Yu, Global stability of a linear nonautonomous delay difference equation, *J. of Difference Eqs. and Appl.* (2), 151–161, (1995).
4. G. Ladas, C. Qian, P.N. Vlahos and J. Yan, Stability of solutions of linear nonautonomous difference equations, *Appl. Anal.* **41**, 183–191, (1991).
5. S. Zhang, Stability of infinite delay difference systems, *Nonlinear Analysis, Theory, Methods and Applications* **22** (9), 1121–1129, (1994).
6. W.F. Ames, *Numerical Methods for Partial Differential Equations*, Academic Press, New York, (1992).

7. W.G. Kelley and A.C. Peterson, *Difference Equations*, Academic Press, New York, (1991).
8. Y.Z. Lin and S.S. Cheng, Stability criteria for two partial difference equations, *Computers Math. Applic.* **32** (7), 87–103, (1996).
9. R. Courant, K. Friedrichs and H. Lewy, On the partial difference equations of mathematical physics, *IBM J.* **11**, 215–234, (1967).
10. X.P. Li, Partial difference equations used in the study of molecular orbits, (in Chinese), *Acta Chim. SINICA* **40**, 688–698, (1982).
11. B.G. Zhang, S.T. Liu and S.S. Cheng, Oscillation of a class of delay partial difference equations, *J. Difference Equations and its Applications* **1**, 215–226, (1995).
12. B.G. Zhang and S.T. Liu, On the oscillation of two partial difference equations, *J. Math. Anal. Appl.* **206**, 480–492, (1997).
13. B.G. Zhang and S.T. Liu, Oscillation of partial difference equations, *PanAmerican Math. J.* **5** (2), 61–70, (1995).
14. B.G. Zhang and S.T. Liu, Necessary and sufficient conditions for oscillations of partial difference equations, *Nonlinear Studies* **3** (2), 187–192, (1996).
15. B. Zhang and C.J. Tian, On the global attractivity of partial difference equations, *J. of Ocean University of Qingdao* (to appear).
16. S.L. Xie and S.S. Cheng, Stability criteria for parabolic type partial difference equations, *J. Comput. Math. Appl.* **75**, 57–66, (1996).
17. G. Berman and K.D. Fryer, *Introduction to Combinatories*, Academic Press, New York, (1972).